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# Droplet theory of correlations in ordered phases

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**Abstract.** The explicit droplet theory of low-dimensional phase transitions, developed recently, is extended to yield a description of the pair correlation function for the universality class of  $q$ -state Potts models in the ordered phase and in zero external field. It is shown that, in  $d = 1 + \varepsilon$  dimensions, the short distance behaviour of the correlation function and, in particular, the exponent  $\eta$ , are controlled by nested nearly spherical droplets. In contrast, it is argued that the spatial dependence of the large distance behaviour, in  $d = 2$  or above, is controlled by highly anisotropic droplets, thus illuminating the Widom relation,  $\sigma\xi^{d-1} \approx \text{constant}$ , linking the dimensionless surface tension  $\sigma$  and the bulk correlation length  $\xi$ . In the particular case of  $d = 2$  the statistical weight of the relevant droplets is determined as that of two appropriately interacting strings, in the spirit of recent, independent, arguments by Abraham and by Fisher: a non-Ornstein–Zernike correlation function prefactor, and the result  $\sigma\xi = \frac{1}{2}$  follow in accord with exact results for the  $q = 2$  (Ising) case, and with implications for the  $q \rightarrow 1$  (percolation) problem.

## 1. Introduction

In a recent paper (Bruce and Wallace 1983, hereafter referred to as I) it was shown how the droplet phenomenology of phase transitions (Fisher 1967 and references therein) can be realised in an explicit theory of Ising systems in  $d = 1 + \varepsilon$  space dimensions. This work has subsequently been extended to include the effects of an ordering field (Sim and Bruce 1984) and generalised to the  $q$ -state Potts universality class (Schmittmann 1982, Schmittmann and Bruce 1984) which incorporates the Ising ( $q = 2$ ) and percolation ( $q \rightarrow 1$ ) problems as special cases.

The present work extends this programme beyond strictly thermodynamic properties to the behaviour of the pair correlation function. In contrast with the bulk of our previous work, the present study leads to some general conclusions and specific results whose validity is not restricted by the  $d = (1 + \varepsilon)$ -dimensional approximation inherent in the framework of our original droplet theory. Nevertheless, since other features of the present study are again subject to this restriction, we shall develop our arguments within the original framework, the elements of which it is thus appropriate to summarise at this juncture.

The foundation stone of the theory is the calculation of the statistical weight for an isolated droplet of one phase embedded in a background of a different phase, in a system of  $q$ -coexisting phases (i.e. a member of the Potts universality class, on its coexistence curve). It was established in I that this single droplet partition function is dominated, for  $d$  near 1, by droplets whose *shape fluctuations* (deviations from spherical) are controllably small, and whose typical linear dimension, or 'scale size',  $R$ , is statistically bounded above by a characteristic length  $\xi$ , which prescribes the

surface tension  $\sigma$  (the free energy per unit interfacial area, expressed in units of  $kT$ ) through a relationship of the form

$$\sigma\xi^{d-1} = c_0^{-1} \quad (1.1)$$

where  $c_0$  is a constant. As a critical temperature  $T_c$  is approached from below, the length  $\xi$  exhibits a power law divergence with a critical index  $\nu$  (whose inverse has a calculable expansion in  $\varepsilon = d - 1$ ), the interfacial free energy vanishes with critical index  $\mu = (d - 1)\nu$ , and the typical droplet scale size diverges. To demonstrate, and to characterise, the implied phase transition (vanishing of long range order) one has to go beyond the isolated droplet theory, and develop a description of a many-droplet assembly, recognising in particular that droplets of one phase may invariably house droplets of the other  $q - 1$  phases. Again we showed that this problem of *droplet nesting* can be handled near  $d = 1$  where droplet boundaries remain controllably dilute even at the critical point. The order parameter of the many-droplet assembly was calculated and found to vanish with a power law; the associated exponent  $\beta$  was found to be related simply ( $\beta = q\psi_0\nu$ ) to a universal parameter  $\psi_0$  prescribed by the form of the single-droplet partition function associated with droplets having scale sizes small compared to  $\xi$ .

The study of the pair correlation function which we present here may most helpfully be subdivided into two parts, which have rather different motivation, offer different insights, and yield results with different domains of validity.

Firstly (§ 2) we will examine the behaviour of the correlation function at distances small compared with  $\xi$ . The motivation here is simply that of completeness: the analysis permits us to identify the remaining fundamental (zero-field) exponent  $\eta$ . The calculation is straightforward: the short distance behaviour is controlled by dilutely nested almost spherical droplets so that the requisite configurational average can be determined by a simple extension of the arguments developed in I. The domain of validity of the argument is, like that of the analysis in I, restricted formally to  $d = 1 + \varepsilon$  dimensions.

The second and more substantial portion of this work is concerned with the behaviour of the correlation function at distances large compared with  $\xi$ . The motivation here is rather more compelling: it is to establish the relationship between the length  $\xi$ , which characterises the critical properties of the interfacial free energy (equation (1.1)) and the true bulk correlation length defining (or defined by) the rate of exponential fall-off which the correlation function is expected to display at large distances. It is generally anticipated that these two lengths will display the same critical behaviour (see e.g. Widom 1972). Indeed this expectation has been built presumptuously into our use of the conventional correlation length and exponent symbols,  $\xi$  and  $\nu$ , for the 'interfacial length' and its critical index. Nevertheless it seems worthwhile to try to justify this presumed equivalence, and to explore its physical origins. This task, addressed in § 3, proves rather more demanding than the first. The difficulty originates in the fact that, in contrast with the thermodynamic properties, considered in I, the large distance behaviour of the correlation function cannot be extracted from a perturbation theory about the spherical droplet limit. Specifically, we shall argue (§ 3.2) that the spatial-dependence of the correlation function for two points separated by a distance  $r \gg \xi$  reflects the  $r$ -dependence of the probability of finding a droplet embracing both points. This conclusion emerges within the framework of our basic droplet model; however it seems likely that its validity transcends the low-dimensional approximation. With this in mind we observe that a spherical droplet fulfilling the two-site overlap constraint will have a surface area of the order of at least

$r^{d-1}$ ; on the other hand the two-site overlap condition may be fulfilled by a highly anisotropic droplet with only one linear dimension of order  $r$ , and the others 'small' compared to  $r$  (in fact, typically of the order of  $\xi$ , although the  $d = 2$  case is exceptional in this regard). Evidently, then, for space dimensions  $d > 2$ , and indeed for  $d = 2$ , the surface energy cost of the anisotropic droplets is small compared to that of the nearly spherical droplets and the former should therefore dominate the large distance ( $r \gg \xi$ ) behaviour of the correlation function. In fact (§ 3.1) we shall see that, at least in  $d = 1 + \varepsilon$  dimensions, the effect of nearly spherical droplets is simply to produce the  $\xi$ -dependent (but  $r$ -independent) prefactor in the correlation function, with the form appropriate for a correctly scaling theory. To capture the  $r$ -dependence of the correlation function one must carefully tailor the analysis according to the (integral!) space dimension of interest. In this paper we shall be predominantly concerned with the large distance behaviour in  $d = 2$ . In this case the relevant anisotropic droplets may be constructed from two strings with (fluctuationless) length  $r$ , tied together by two further strings with lengths of the order of  $\xi$ . This picture is essentially that envisaged in recent independent studies by Abraham (1983) and Fisher (1984) and the key results we shall obtain in this context are already apparent in these two papers. Specifically we shall see (§ 3.3) that the string-based theory accounts for the non-Ornstein-Zernike ( $r^{-2}$ ) form of prefactor long known to characterise the ordered phase of the  $d = 2$ ,  $q = 2$  (Ising) lattice model. We shall also see that equation (1.1) is indeed satisfied in  $d = 2$  with  $\xi$  the bulk correlation length, and with the constant  $c_0$  assigned the value  $c_0 = 2$ , so that

$$\sigma\xi = \frac{1}{2} \tag{1.2}$$

which is, once again, a result that is known to hold exactly for the  $d = 2$  Ising model.

Although these results are, then, in no sense new, the present exercise has some potentially informative features. Firstly, since it is technically different from the analyses of Abraham (1983) and Fisher (1984), its success serves to confirm that the 'understanding' of the aforementioned results afforded by the string picture is not illusory. Secondly, the methods developed and shown to work successfully here are potentially applicable to the large distance behaviour of the  $d = 3$  ordered phase correlation function, whose prefactor structure is not known. Thirdly we shall argue that the two key features of the  $d = 2$  correlation function identified above may well be characteristic not just of the Ising problem but, more generally, of the  $q$ -state Potts universality class (or, at least those members of this class which exhibit continuous phase transitions). In particular, in the  $q \rightarrow 1$  limit which describes the percolation problem (Fortuin and Kasteleyn 1972) the relationship (1.2) prescribes a definite link between the pair connectedness length and the length which controls the exponential decay of the cluster number distribution. These remarks are amplified in our concluding and summarising section.

## 2. The short distance behaviour of the correlation function

### 2.1. The configuration-generating procedure

In I we devised a procedure by which one may generate a representation of the canonical ensemble of configurations in an Ising system, in  $d = 1 + \varepsilon$  dimensions. This procedure was subsequently extended to the Potts universality class (Schmittmann

1982). We begin our analysis by recalling this procedure, in outline only; the reader is referred to I for a fuller exposition and to subsequent papers (Sim and Bruce 1984, Schmittmann and Bruce 1984) where the procedure is developed in a more explicit and rather more conventional renormalisation group format.

The method rests on the key result of I, expressing the partition function for a single droplet of one phase embedded in a volume  $V$  of a different phase (equation I, 4.1)

$$Z_1 = V \int dR v^{-1}(R) \psi(R, \xi) \quad (2.1)$$

where  $v(R) = d^{-1} S_d R^d$  is the volume of a sphere of radius  $R$  and  $\psi(R, \xi)$  is a function with the following properties

$$\psi(R, \xi) = R^{-1} \tilde{\psi}(R/\xi) \quad (2.2a)$$

$$\tilde{\psi}(z) \begin{cases} \approx \psi_0 & (z \ll 1) \\ \sim z^{(d^2-1)/2} \exp(-S_d z^{d-1}/c_0) & (z \gg 1). \end{cases} \quad (2.2b)$$

$$(2.2c)$$

The universal parameter  $\psi_0$  is defined in equation (I, 4.4d). The critical length  $\xi$  is defined in equation (I, 3.28b). Within the context of the theory this length effectively parameterises the surface tension. Thus the argument of the exponential in (2.2c) is recognisable (given equation (1.1)) as the interfacial free energy of a spherical droplet of radius  $R$ . We shall subsequently see that, with appropriate identification of the parameter  $c_0$  (which is *not* prescribed by the arguments of I), the critical length  $\xi$  does indeed coincide with the true bulk correlation length.

In § 5.2 of I it was shown that the function  $\psi(R, \xi) dR$  has a simple physical interpretation: it represents the mean fraction of the available space of one phase occupied by droplets of another phase, having scale sizes in the range  $R$  to  $R + dR$ . This assertion is embodied and refined in the procedure by which, it was argued, one may generate an approximate representation of the configurations in a  $q$ -phase assembly, allowing for the two complementary effects of excluded volume and droplet nesting. Specifically we decorate an ensemble of macroscopically-sized samples of one particular phase with successively smaller droplets of all  $q$  phases; at each stage of this differential dressing procedure we use droplets of each phase and of the scale size  $R \rightarrow R + dR$  then appropriate, in such numbers as to cover a mean fraction  $\psi(R, \xi) dR$  of the existing available space (i.e. the space occupied by the other phases); the dressing procedure, which is realised in the form of appropriate differential equations, is terminated when the decorating-droplet scale size reaches some arbitrary minimum length  $L_0$ . The latter coordinate, and the scale of the ordering coordinate together define the two basic non-universal scales appearing in any strong scaling theory; modulo these two scales the observables calculated by appropriate averages over the prescribed ensemble should display the universal characteristics of the class of  $q$ -state Potts systems.

## 2.2. Application to the correlation function

Consider an assembly decorated in the fashion described above with droplets of  $q$  phases, of scale sizes larger than  $L_0$ , and characterised by a particular critical length scale  $\xi$ , effectively defining the assembly temperature. Of the  $q$ -phases let phase 1 be that which is thermodynamically favoured (by the boundary conditions realised in our

framework through the choice of the phase of the undecorated assembly). We define an order parameter  $Q$  for the assembly by

$$Q(L_0, \xi) = q(q-1)^{-1}\{p_1(L_0, \xi) - q^{-1}\} \tag{2.3}$$

where  $p_1(L_0, \xi)$  is the probability that a point in the assembly lies in phase 1. In the same fashion we define a pair correlation function  $g$  by

$$g(r; L_0, \xi) = q(q-1)^{-1}\{p_s(r; L_0, \xi) - q^{-1}\} \tag{2.4}$$

where  $p_s(r; L_0, \xi)$  is the probability that two points separated by distance  $r$  lie in the *same* phase. With these conventions it is easily seen that

$$\lim_{r/\xi \rightarrow \infty} g(r; L_0, \xi) = Q^2(L_0, \xi) \tag{2.5}$$

so that the appropriate connected pair correlation function is simply

$$\hat{g}(r; L_0, \xi) = g(r; L_0, \xi) - Q^2(L_0, \xi). \tag{2.6}$$

Now, in the spirit of the arguments of I, we set up a differential equation describing the manner in which the probability  $p_s$  evolves as one changes the inner length scale from  $L_0$  to  $L_0 - |dL_0|$  by decorating the assembly with droplets whose scale size lies in this infinitesimally narrow range. In so doing we shall restrict ourselves to the regime in which the dressing droplet size  $L_0$  is small enough compared to the site separation  $r$  that droplets of this scale size have negligible probability of overlapping both sites. Then one readily sees that

$$dp_s(r, L_0, \xi) = \psi(L_0, \xi)|dL_0|\{-2(q-1)p_s(r; L_0, \xi) + 2(1 - p_s(r; L_0, \xi))\}. \tag{2.7}$$

The first term in the parentheses represents the effects of those configurations in which one of the two locations (separated by  $r$ ) initially occupying the *same* phase is subsequently covered by one of  $q-1$  types of dressing droplet; the second term represents the effects of configurations in which one or other of two locations initially occupying *different* phases is subsequently covered by a dressing droplet of the phase in which the other site is embedded. Combining equations (2.4) and (2.7) we find immediately that the pair correlation function satisfies the differential equation

$$dg(r; L_0, \xi)/dL_0 = 2q\psi(L_0, \xi)g(r; L_0, \xi), \quad L_0 \ll r. \tag{2.8a}$$

On the other hand we may recall from our earlier studies (or recover directly from (2.3)) that the order parameter satisfies the differential equation

$$dQ^2(L_0, \xi)/dL_0 = 2q\psi(L_0, \xi)Q^2(L_0, \xi). \tag{2.8b}$$

Combining equations (2.8a), (2.8b) and (2.6), and integrating the resulting equation for the connected correlation function we find

$$\hat{g}(r; L_0, \xi) = \hat{g}(r; L_m, \xi) \exp\left(-2q \int_{L_0}^{L_m} \psi(R, \xi) dR\right) \tag{2.9}$$

where, for consistency (cf our neglect of two site overlap) we require that our ‘matching’ length  $L_m$  must be small compared to the site separation  $r$ . Choosing  $L_m = \alpha r$ , with  $\alpha \ll 1$ , assuming that  $r$  is small compared to  $\xi$  and invoking the small  $R/\xi$  behaviour of the function  $\psi$  (equations (2.2a, b)) we find

$$\hat{g}(r; L_0, \xi) = D(\alpha)(L_0/r\alpha)^{2q\psi_0} \tag{2.10a}$$

with

$$D(\alpha) = \lim_{\xi \rightarrow \infty} \hat{g}(r; \alpha r, \xi). \quad (2.10b)$$

On dimensional grounds it is clear that this amplitude is independent of  $r$ ; consistency requires that it should depend upon  $\alpha$  in such a way as to cancel the explicit  $\alpha$ -dependence in (2.10a). Thus we conclude that

$$\hat{g}(r; L_0, \xi) = D(L_0/r)^{d-2+\eta} \quad (2.11a)$$

with  $D = D(1)$  and

$$\eta = 2 - d + 2q\psi_0 = 2 - d + 2\beta/\nu \quad (2.11b)$$

in accord with strong scaling.

### 3. The large distance behaviour of the correlation function

#### 3.1. General scaling theory

We begin by returning to equation (2.9). This result is valid as long as the matching length  $L_m$  is small compared with  $r$ . In the regime  $r \gg \xi$ , with which the remainder of this paper is concerned, we may satisfy the former constraint ( $L_m \ll r$ ) with a matching length which is nevertheless large compared to the correlation length ( $L_m \gg \xi$ ). Let us consider the implications of this choice for the correlation function  $\hat{g}(r; L_m, \xi)$  which appears on the RHS of equation (2.9). This function describes the correlations in an assembly from which all droplets of scale sizes  $R < L_m$  have been erased. The choice  $L_m \gg \xi$  then effectively ensures that this assembly is non-critical; more precisely it implies that those droplets which may appear in the assembly will do so with such small probability that their nesting and excluded volume may be neglected, thus realising the situation envisaged in the original droplet model (Fisher 1967). In effect the argument leading to equation (2.9) fulfils the central aim of any renormalisation group analysis (and such, indeed, it is) of relating the intractable physics of a near-critical assembly to the hopefully simple properties of an assembly far from criticality.

Exploiting the simplicity of the dilute limit we may now write

$$p_1(L_m, \xi) \approx 1 - (q-1)p_c(L_m, \xi) \quad (3.1a)$$

where  $p_c(L_m, \xi)$  is the probability that a particular site is covered by some droplet of a particular one of the  $q-1$  unfavoured phases. Furthermore

$$p_s(r; L_m, \xi) \approx 1 - 2(q-1)p_c(L_m, \xi) + 2(q-1)p_b(r; L_m, \xi) \quad (3.1b)$$

where  $p_b(r; L_m, \xi)$  is the probability that the two sites are both covered by the *same* droplet of a particular one of the  $q-1$  unfavoured phases. Combining equations (2.3), (2.4), (2.6) and (3.1) we find

$$\hat{g}(r; L_m, \xi) \approx 2qp_b(r; L_m, \xi) \quad (3.2)$$

whence, from equation (2.9), we obtain

$$\hat{g}(r; L_0, \xi) \approx 2qp_b(r; L_m, \xi) \exp\left(-2q \int_{L_0}^{L_m} \psi(R, \xi) dR\right). \quad (3.3)$$

The argument is completed by two further observations. Firstly, given the condition

$L_m \gg \xi$  (which we have chosen to fulfil), the exponential on the RHS of equation (3.3) is asymptotically independent of  $L_m$  and is recognisable (cf equation (2.8*b*)) as the square of the bulk order parameter:

$$\exp\left(-2q \int_{L_0}^{L_m} \psi(R, \xi) dR\right) \approx Q^2(L_0, \xi) \quad L_m \gg \xi \quad (3.4a)$$

Secondly, given the condition  $r \gg L_m$  (which we have also chosen to fulfil), the probability  $p_b$  should clearly (both on intrinsic physical grounds, and for the sake of the  $L_m$ -independence of the RHS of equation (3.3)) be independent of  $L_m$  and thus be a scaling function of the ratio  $r/\xi$ :

$$p_b(r; L_m, \xi) \approx \tilde{p}_b(r/\xi) \quad r \gg L_m. \quad (3.4b)$$

Combining these results we find

$$\hat{g}(r; L_0, \xi) \approx 2qQ^2(L_0, \xi)\tilde{p}_b(r/\xi) \quad r \gg \xi \quad (3.5a)$$

which has the anticipated scaling structure

$$\hat{g}(r; L_0, \xi) \approx \xi^{-2\beta/\nu} \tilde{g}(r/\xi). \quad (3.5b)$$

Equation (3.5*a*) is the key result of this paper. It has implications for the  $r$ -dependence, the  $\xi$ -dependence and the  $q$ -dependence of the two-point correlation function at large distances. We defer discussion of the  $\xi$ - and  $q$ -dependence to § 4. Here we shall be primarily concerned with the implied spatial dependence which, according to equation (3.5*a*), should be controlled by the spatial-dependence of the probability of finding a closed interface (droplet) embracing the two points. We have of course established this result only within the dilute droplet boundary approximation justifiable formally in  $d = 1 + \varepsilon$  dimensions. However we shall proceed on the assumption that its domain of validity is actually not restricted to dimensions  $d$  'close' to 1. By the time we return to consider this assumption, in § 4, we shall have accumulated considerable circumstantial evidence to support it.

### 3.2. *The two-site overlap probability; general remarks*

Before attempting to evaluate the two-site overlap probability it is essential to identify the qualitative character of the droplet configurations which control its behaviour. Given the prevailing philosophy of our basic droplet theory one might naturally look first to the contribution made by nearly spherical droplets of scale size  $R > \frac{1}{2}r$ . One immediately discovers, however, that the interfacial free energy cost of such droplets will ensure that their statistical weight will fall off with  $r$  as

$$\exp[-\sigma S_d (r/2)^{d-1}] \sim \exp[-(S_d/c_0)(r/2\xi)^{d-1}] \quad (3.6)$$

where we have anticipated the relationship (1.1). For  $d > 2$  the spatial dependence of (3.6) is unambiguously different from the anticipated simple exponential behaviour. In contrast to suggestions by Reatto and Rastelli (1972) and Stauffer (1978) the author does not believe that this difficulty can be circumvented by invoking the finite interfacial thickness of droplet boundaries, realised within our framework as fluctuation-induced distortions about the spherical limit. Instead we note that, whereas one linear dimension of the relevant droplets must exceed  $r$ , the others need not. Consider then the interfacial free energy cost of an anisotropic droplet formed from a cylinder of length  $r$  and



radius  $R$ . This free energy will be of the order of  $\sigma_{\text{eff}} R^{d-2} r$  where  $\sigma_{\text{eff}}$  is an effective surface tension. In the limit in which  $r$  and  $R$  are both large compared to  $\xi$  it is clear that  $\sigma_{\text{eff}}$  should coincide with the macroscopic surface tension  $\sigma$ . Allowing for arbitrary  $R$  values, but taking  $r \gg \xi$ , consistent with our basic premise, one may anticipate that  $\sigma_{\text{eff}}$  will be independent of  $r$  and thus, on dimensional grounds, should have the crossover scaling form.

$$\sigma_{\text{eff}}(R, \xi) = R^{-(d-1)} \tilde{\sigma}(R/\xi) \quad (3.7a)$$

where

$$\tilde{\sigma}(z) \sim \begin{cases} z^{d-1} & z \gg 1 \\ \text{constant} & z \ll 1 \end{cases} \quad (3.7b)$$

The free energy cost of the droplet considered is then of the form

$$\sigma_{\text{eff}} R^{d-2} r = (r/\xi) \cdot (\xi/R) \tilde{\sigma}(R/\xi). \quad (3.8)$$

Inspection of (3.8) shows, then, that provided the 'constant' appearing in (3.7b) is non-zero the free energy cost of such droplets will be minimal for  $R$  values of the order of  $\xi$  and will have the order of magnitude

$$\sigma_{\text{eff}} R^{d-2} r \sim r/\xi. \quad (3.9)$$

This is, in fact, the result one obtains if one boldly utilises the *macroscopic* surface tension, together with the Widom relation (1.1), to estimate the free energy of 'cylindrical' droplets of radius  $\sim \xi$ . The fact that the spatial dependence it implies for the correlation function coincides with the anticipated simple exponential form may, then, according to one's point of view, be taken either as an alternative argument for the Widom relationship, or as a vindication of the conjecture that such highly anisotropic droplets do indeed dominate the large distance behaviour of the ordered phase correlation function in  $d > 2$ . (The  $d = 2$  case is actually somewhat special, as we shall see.)

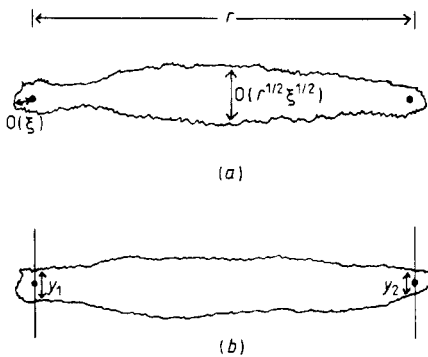
In the light of these arguments it is clear that the problem of the ordered phase correlation function can certainly not reasonably be tackled with perturbative expansions about the spherical droplet limit—at least, not in any space dimensions of physical interest. Rather one must tailor the analysis in such a way that any approximations made concerning fluctuations are acceptable, specifically, for those (highly anisotropic) configurations which one expects to be dominant. In the rest of this paper we proceed to examine the  $d = 2$  case in some detail. In so doing we relinquish the formally firm but practically restrictive foundation for the argument leading to equation (3.5a). Nevertheless we shall accumulate at the very least pragmatic justification for our faith in the more general validity of this result, and in so doing provide motivation for studying its consequences for the three-dimensional problem.

### 3.3. The two-site overlap probability in $d = 2$

**3.3.1. The two-string picture: the surface tension and the correlation length.** The behaviour of the correlation function in the ordered phase of the  $d = 2$ ,  $q = 2$  (Ising) universality class is now both well established and well understood. Firstly there exist exact calculations for the nearest-neighbour lattice Ising model (Wu 1966, McCoy and Wu 1973). Secondly, in papers received during the course of the studies reported here, Abraham (1983) and Fisher (1984) have shown that the essential features of the large

distance behaviour displayed by the exact results can indeed be understood in terms of the properties of highly anisotropic droplets. This perspective is developed here using methods which, it is hoped, are complementary to those deployed in the above noted papers and which offer the possibility of tackling the  $d = 3$  problem.

In  $d = 2$  the argument developed in the preceding section requires a little modification and can, in fact, be reformulated in a more powerfully predictive form. Certainly one may still anticipate that the relevant droplets will be highly anisotropic, having 'lengths' (of order  $r$ ) large compared to their 'widths'. However, by comparison with the  $d > 2$  situation the transverse dimension (at points intermediate between the two sites) affects the interfacial energy cost only weakly, as long as it is small compared to  $r$ , and the interfaces will thus wander more widely. This entropy-driven wandering will be limited by the requirement that the interfaces do not cross, *and* by the interfacial energy cost of the droplet 'ends' (which would appear to have a negligible role in higher dimensions). These considerations suggest that the typical contributing configuration will have the form suggested schematically in figure (1a). This picture embodies the essential ingredients of the arguments of Abraham (1983) and Fisher (1984). The typical relevant droplet may be regarded as being constructed from two strings of fluctuationless (minimal) length  $r$ . The constraint that the two strings should pass through the two points in question (imposed in the aforementioned papers) is not strictly justifiable and, more generally, one can regard the two  $r$ -lengthed strings as being joined at each end by two further strings, in the manner suggested in figure (1b). It is clear, both from the perspectives of general scaling theory, and from the more specific implications of equation (1.1), that the end strings will have lengths of the order of  $\xi$ .



**Figure 1.** (a) Schematic representation of a droplet contributing to the two point function at distance  $r \gg \xi$  in the ordered phase in two dimensions. (b) The four-string representation of a contributing droplet, analysed in the text.

That the gross features of this picture are correct is made clear by its capacity to predict (and in the process illuminate) the relationship between the surface tension and the correlation length, known to hold for the  $d = 2$  Ising model. Specifically, we see that the interfacial free energy cost of the anisotropic droplets in question will (for  $r$  large compared to  $\xi$ ) approach  $2\sigma r$ . The associated statistical weight of such configurations will thus fall off, with  $r$ , as  $e^{-2\sigma r}$  (to within pre-exponential factors to be considered shortly). With the perspectives of equation (3.5a), this exponential

dependence must match that of the pair correlation function itself, so that

$$e^{-2\sigma r} \sim e^{-r/\xi} \tag{3.10}$$

from which one immediately infers equation (1.2). The Widom relation (1.1) is thus vindicated and the universal constant  $c_0$  appropriate for  $d = 2$  is identified as  $c_0 = 2$ . Alternatively, approaching the problem from the perspectives of our original droplet theory, one may assert that the ‘critical length scale’ which parameterises the surface tension in the manner indicated by equation (2.2c) may be brought into coincidence with the bulk correlation length by the choice  $c_0 = 2$ . (Strictly, to complete this argument one must show that the surface tension inferred from the statistical weight of large spherical droplets, equation (2.2c), is the same as that characterising a planar interface. This has indeed been checked through studies of the planar interface using the same form of dimensionally-regularised renormalisation group arguments as utilised in the study of nearly spherical droplets in  $d = 1 + \epsilon$ . The details will not be given here.)

Gratifyingly, as pointed out by Fisher (1984) equation (1.2) actually expresses an exact result for the ordered phase of the  $d = 2$  lattice Ising model (holding even when the surface tension and correlation length are anisotropic quantities). Within the framework made plausible by the arguments of § 3.1, we conjecture that this result should in fact hold for the class of  $q$ -state Potts systems (cf further discussion in § 4). For the present, however, we proceed to put the picture advanced in figure 1(a) to a more stringent test by exploring its implications for the subdominant  $r$ -dependent prefactor of the correlation function.

*3.3.2. The single-string partition function.* It will prove helpful to preface the calculation of the statistical weight of the (essentially) two-string configurations depicted in figure 1(a) with an analysis of a ‘single-string’ problem, along the lines of an argument given by Stack and Stone (1981).

Consider a string of minimal (fluctuationless) length  $r$  fixed at each end, but otherwise free to fluctuate into the  $d - 1$  dimensions orthogonal to its length. The actual length of any configuration may be written in the form

$$L_d(\{f\}) = \int_0^r dx \left( 1 + \sum_{\alpha=1}^{d-1} \left( \frac{\partial f^\alpha(x)}{\partial x} \right)^2 \right)^{1/2} \tag{3.11}$$

where  $f^\alpha(x)$  is the  $\alpha$ th component of the  $(d - 1)$ -dimensional vector  $f(x)$  giving the transverse displacement (from the fluctuationless state) of the string at point  $x$ . We associate with the spectrum of single-string configurations a statistical weight

$$Z_{1S}(r) = \int Df \exp\{-\sigma_0 L_d(\{f\})\} \tag{3.12}$$

where  $\sigma_0$  represents a bare string tension. This equation is meaningful only once we have assigned a precise significance to the functional integral. We shall make the Fourier decomposition

$$f(x) = a_0 \left( \frac{2}{N} \right)^{1/2} \sum_{n=1}^{N-1} b(n) \sin(n\pi x/r) \tag{3.13a}$$

with

$$N = r/a_0 \tag{3.13b}$$

and adopt the convention

$$\int Df = \int \prod_{n=1}^{N-1} d\mathbf{b}(n). \tag{3.14}$$

These assignments are consistent with (and motivated by) the discretised (lattice) form of (3.12) considered in appendix 2. They are not unique: the results we shall obtain are thus trustworthy only to the extent that they are independent of the details of this choice.

Proceeding with this in mind we expand the configurational length (3.11) in powers of  $f$ , and utilise the representations (3.13*a, b*) to find

$$\begin{aligned} Z_{1S}(r) &= \int \prod_{n=1}^{N-1} d\mathbf{b}(n) \exp\left(-\sigma_0 r - \frac{\sigma_0 a_0^3 \pi^2}{2r^2} \sum_{n=1}^{N-1} n^2 b^2(n) + O(b^4)\right) \\ &= \exp\left(-\sigma_0 r - \frac{(d-1)}{2} \sum_{n=1}^{N-1} \ln\left(\frac{n^2 \sigma_0 a_0^3 \pi}{2r^2}\right)\right). \end{aligned} \tag{3.15}$$

To the requisite accuracy we find for the remaining sum

$$\begin{aligned} \sum_{n=1}^{N-1} \ln\left(\frac{n^2 \sigma_0 a_0^3 \pi}{2r^2}\right) &= (N-1) \ln\left(\frac{\sigma_0 a_0^3 \pi}{2r^2}\right) + 2(N-\frac{1}{2}) \ln N - 2N + O(N^0) \\ &= \frac{r}{a_0} [\ln(\sigma_0 a_0 \pi / 2) - 2] + \ln(r/a_0) + O(1) \end{aligned} \tag{3.16}$$

whence

$$\begin{aligned} Z_{1S}(r) &= \exp\left\{-\sigma_0 r \left[1 + \frac{(d-1)}{2\sigma_0 a_0} [\ln(\sigma_0 a_0 \pi / 2) - 2] + O(\sigma_0^{-2} a_0^{-2})\right]\right. \\ &\quad \left. - \left(\frac{d-1}{2}\right) \ln\left(\frac{r}{\sigma_0 a_0^2}\right) + O(1)\right\} \\ &= r^{-(d-1)/2} z(\sigma, a_0) e^{-\sigma r}. \end{aligned} \tag{3.17}$$

The last step requires some elaboration. Firstly, we have identified the terms in the coefficient of  $r$  in the exponential as contributions to the fluctuation-renormalisation of the bare string tension. The naive perturbation theory in  $(\sigma_0 a_0)^{-1}$  does not now allow us to control these fluctuation effects in a systematic way. However these effects are not now the focus of concern since they ultimately serve only to determine the critical behaviour of the interfacial tension of our ( $d=2$ ) anisotropic droplets, and thence the coefficient of  $r$  in the dominant exponential factor in the correlation function, discussed in the preceding section. Secondly, we have subsumed into an unspecified prefactor  $z$  those logarithmic terms in the exponential which do not involve  $r$  (either as a coefficient or as an argument). The specific form of this prefactor emerging from the present calculation is not trustworthy in as much as it depends upon the details of the way in which the Fourier sum (3.13*a*) is truncated. Indeed one might feel that the prefactor is inherently ill defined given the ambiguity associated with the choice of measure (3.14). In fact it seems likely that the prefactor *can* be assigned an absolute significance, just as (cf I) it appears that there is an absolute measure for the partition function of isolated droplets. We shall not, however, explore this problem here since the present exercise is designed to capture and illuminate the  $r$ -dependence (rather than the  $\xi$ -dependence) of the correlation function prefactor. To that end we note

now the power of  $r$  which appears in the prefactor of (3.17) and which can be traced to the coefficient of the logarithm of  $r$  in the regularised sum (3.16). This power is, we believe, trustworthy, since it is quite insensitive to the specific way in which the Fourier sum (3.13a) is truncated, and since higher-order terms in the perturbation theory do not appear to generate additional logarithmic factors. The particular form of  $r$ -dependent prefactor structure implied by equation (3.17) is recognisable as that of the Ornstein-Zernike prefactor of the disordered phase correlation function, as pointed out by Stack and Stone (1981). This correspondence is immediately intelligible in the light of the graphical representation of high-temperature series expansions for the two-point correlation function which takes the form of a sum over self-avoiding strings of bonds running between the two points (Fisher and Burford 1967). The coincidence of the two results suggests that the prefactor structure is not influenced by the string overhangs and separate closed strings present in the series expansion formalism, but not contained in the configurational sum (3.12).

*3.3.3. The two-string partition function, and the correlation function prefactor.* We now turn to the two-string problem. Rather than confront directly the spectrum of configurations indicated in figure 1(b) we shall consider in the first instance a slightly simpler situation: two strings are pinned at each end a distance  $y$  apart, but are otherwise free to fluctuate in a two-dimensional space; we suppose that the strings experience a short-range repulsive interaction  $V$ , which suppresses string crossing (and may eliminate it altogether if taken to the infinite strength limit). We write the associated configurational weight in the form

$$Z_{2S}(r, y) = \int Df_1 \int Df_2 \exp\left(-\sigma_0[L_2(\{f_1\}) + L_2(\{f_2\})] - \int_0^r dx V[y + f_2(x) - f_1(x)]\right) \quad (3.18)$$

where  $f_1$  and  $f_2$  denote the displacement fields of the strings from their minimum length configurations. Expanding the configurational lengths in powers of the fields to leading order, and replacing the two field variables by centre of mass and relative separation coordinates we find that

$$Z_{2S}(r, y) = Z_{1S}(r)Z_{2R}(r, \tilde{y}) \quad (3.19a)$$

where

$$Z_{2R}(r, \tilde{y}) = \int Df \exp\left(-\sigma_0 r - \int_0^r dx [\frac{1}{2}\sigma_0(\nabla f(x))^2 + \tilde{V}(\tilde{y} - f(x))]\right). \quad (3.19b)$$

The single-string configurational weight appearing in equation (3.19a) originates in the functional integral associated with the centre of mass coordinates. The remaining functional integral  $Z_{2R}$  reflects the contribution of the relative coordinates; we have defined  $\tilde{y} = y/\sqrt{2}$  and  $\tilde{V}(u) = V(\sqrt{2}u)$ , while the boundary conditions require that  $f(0) = f(r) = 0$ . This functional integral may be evaluated by transfer operator methods similar to those used by Abraham (1983), who treats the analogous solid-on-solid problem; we summarise this argument in the appendix. Here we give a simple argument which serves to capture the result of the complete analysis, and illuminate it somewhat.

Let us introduce a restricted single-string configurational weight

$$Z_{1S}(r, \tilde{y}) \int_{>} Df \exp\{-\sigma_0 L_2(\{f\})\} \quad (3.20)$$

where the functional integral  $\int_{>}$  extends only over those configurations which reach *beyond* a line  $f(x) = \tilde{y}$  (i.e. configurations for which  $f(x) > \tilde{y}$  for some  $x$ ). Clearly

$$Z_{1S}(r, \tilde{y} = 0) = Z_{1S}(r) \tag{3.21a}$$

while, in the limit of sufficiently strong repulsive potentials  $\tilde{V}$  (to which regime this argument is restricted), one must have

$$Z_{2R}(r, \tilde{y}) = Z_{1S}(r, \tilde{y} = 0) - Z_{1S}(r, \tilde{y}) \tag{3.21b}$$

since the configurational integral over the relative coordinates in  $Z_{2R}$  includes (in this limit) only configurations which do *not* reach beyond the line  $f(x) = \tilde{y}$ . Now, rather generally we anticipate that  $Z_{1S}(r, \tilde{y})$  will have the form

$$Z_{1S}(r, \tilde{y}) = \exp\{-rG_0(\tilde{y}/r) - \ln r \cdot G_1(\tilde{y}/r) + O(r^0)\} \tag{3.22}$$

or, in view of (3.21a),

$$Z_{1S}(r, \tilde{y}) = Z_{1S}(r) \exp\{-r[G_0(\tilde{y}/r) - G_0(0)] - \ln r[G_1(\tilde{y}/r) - G_1(0)]\}. \tag{3.23}$$

Now the product  $rG_0(\tilde{y}/r)$  is simply the (dimensionless) free energy of the interface *constrained* by the requirement that  $f(x) > \tilde{y}$ , for some  $x$ . Thus we expect that

$$rG_0(\tilde{y}/r) = \sigma L_{\min}(r, \tilde{y}) \tag{3.24}$$

where  $L_{\min}(r, \tilde{y})$  is the minimum interfacial length consistent with the constraints. Clearly

$$L_{\min}(r, \tilde{y}) = 2[(r/2)^2 + \tilde{y}^2]^{1/2} = r + 2\tilde{y}^2/r + O(\tilde{y}^4/r^3) \tag{3.25}$$

whence

$$Z_{1S}(r, \tilde{y}) = Z_{1S}(r) \exp(-2\sigma\tilde{y}^2/r)[1 + O(1/r^2)] \tag{3.26}$$

where in the last step we have used the fact that the leading  $\tilde{y}$  dependence of the function  $G_1$  must be at least of order  $\tilde{y}^2$ . Combining equations (3.21) and (3.26) we find

$$\begin{aligned} Z_{2R}(r, \tilde{y}) &= Z_{1S}(r)(1 - \exp(-2\sigma\tilde{y}^2/r)) \\ &\approx (2\sigma\tilde{y}^2/r)Z_{1S}(r) \end{aligned} \tag{3.27}$$

in the regime  $\sigma\tilde{y}^2 \ll r$  which will be of importance. This result is essentially that obtained by Camp and Fisher (1972), using the method of 'images', as the generalisation of Ornstein-Zernike theory to the description of correlations in the presence of an absorbing boundary. Now, recalling (3.19a) we find

$$Z_{2S}(r, y) \approx (\sigma y^2/r)Z_{1S}^2(r). \tag{3.28a}$$

It is not hard to generalise the analysis to the situation typical of the configurations alluded to in figure 1(b) where the separations of the ends of the string are different,  $y_1$  and  $y_2$  say. The result (derived by transfer matrix in the appendix) is simply

$$Z_{2S}(r, y_1, y_2) = (\sigma y_1 y_2 / r) Z_{1S}^2(r) \tag{3.28b}$$

in the limit in which the  $y$  variables are of the order of  $\xi$  or less.

The form of the two-site overlap probability now follows readily as the integral over all configurations of the type depicted in figure 1(b):

$$\tilde{p}_b(r/\xi) \sim \int_0^\infty dy_1 \int_0^\infty dy_2 Z_{2S}(r, y_1, y_2) Z_{1S}(y_1) Z_{1S}(y_2). \tag{3.29}$$

The equivalence should hold to within  $r$ -independent quantities dependent upon the measures of the various functional integrals we have calculated, and the measures of the  $y$ -integrals themselves. The two single-string partition functions reflect the configurational weight of the end strings. It is clear that their effect is to cut off the integrand in (3.29) at values of  $y_1$  and  $y_2$  of the order of  $\xi$ , and thus within the range in which the small  $y_1$  and  $y_2$  expansion leading to equation (3.28*b*) is valid. Physically, then, the typical 'width' of the droplets concerned, in the vicinity of the end points is controlled by the interfacial energies of the 'end' strings rather than the energies of the two main ( $\sim r$ -lengthed) strings. The essential structure of the two-site overlap probability then follows as (cf equations 3.17 and 3.28*b*)

$$\tilde{p}_b(r/\xi) \sim r^{-2} e^{-r/\xi} \quad (3.30)$$

where we have fed in the implications of (3.10). As one would hope, in the light of equation (3.5*a*), the  $r$ -dependent prefactor is in accord with the exactly established behaviour of the correlation function of the nearest-neighbour lattice Ising model for  $T < T_c$  (Wu 1966). Its origins, and its generality, are exposed by the arguments by which it has been established here, as we now finally turn to discuss.

#### 4. Discussion and summary

In this concluding section we shall review the results we have obtained, commenting on the insights which they offer, on their trustworthiness, on their implications in a number of contexts and on the possibilities for future extensions.

Our study of the short distance behaviour of the correlation function (§ 2) represents a straightforward extension of the nested nearly spherical droplet theory developed in I. As such it is formally justifiable only in  $d = 1 + \varepsilon$  dimensions; however, in view of the smallness of the known value of  $\beta/2\nu$  (corresponding to  $\psi_0$ , the 'small parameter' of the theory) in  $d = 2$ , it seems reasonable to expect that the underlying picture is substantially correct in two dimensions. In any event the wholly unsurprising identification (2.11*b*) shows that  $\eta$ , like  $\beta$ , has an essential singularity in the  $\varepsilon \rightarrow 0$  limit.

By contrast our study of the large distance behaviour of the correlation function (§ 3) transcends the  $(1 + \varepsilon)$ -dimensional framework of our original droplet theory in a number of respects and, in the process, seems to gain in richness at least as much as it loses in rigour. The key result underpinning and motivating this, the bulk of our study, is equation (3.5*a*). Although this result has been derived within the context of the dilutely nested droplet theory it seems quite likely that it is actually rather more generally valid. In exploring this proposition we must deal separately with the  $r$ -dependence implied by (3.5*a*), the residual  $\xi$ -dependence embodied in the order parameter prefactor, and the nature of the implied  $q$ -dependence.

That the spatial dependence of the two-point correlation function should reflect the probability of finding a closed interface surrounding the two points is plausible on at least two counts. Firstly, as noted by Fisher (1984), it is consistent with the picture suggested by low-temperature series expansion studies (Tarko and Fisher 1975) in which graphs contributing to the correlation function are found to involve connected clusters of spins (overturned with respect to the ground state) embracing the two sites. Secondly, there is clearly considerable circumstantial support for the spatial dependence implied by (3.5*a*) to be found in the successes which issue from this prediction. It leads directly to the Widom relation (1.1) in general dimension  $d > 2$ , to the specific

and simple form this relationship assumes in  $d = 2$  (equation 1.2), and even to the correct  $r$ -dependent prefactor in  $d = 2$ , which we shall be discussing at length below.

The question of the reliability of the  $\xi$ -dependent prefactor suggested by (3.5a) is more subtle. Certainly, in as much as it bears out scaling predictions the actual power of  $\xi$  (or reduced temperature) embodied in the order parameter prefactor is indeed generally correct. However the predictive power claimed by equation (3.5a) is, possibly, greater than this, since it purports to capture the overall scale of the correlation function. Of course the scale of the 'prefactor' in (3.5a) is meaningful only to the extent that the overall scale of the two-site overlap probability may itself be assigned a precise significance. With one caveat this does indeed appear to be possible in principle. It would, however, be hard in practice since it would involve careful consideration of the measures of the functional integrals determining this probability; although it seems likely that these measures can be assigned unambiguous significance within the framework of dimensional regularisation (cf I appendix 1) it is unlikely that the results would be trustworthy when extrapolated to interesting dimensions. Even in principle, however, this procedure would appear to be meaningful only to the extent that one may treat the interface of the relevant droplet as effectively isolated; this is of course, strictly, the precondition for the validity of equation (3.5a). Thus one is still left with the possibility that equation (3.5a) is too naive in its implications that critical fluctuations merely produce essentially separate renormalisations of the bulk (ordering coordinate scale) and surface (interfacial tension) properties of the droplets. There are, however, circumstances in which it is now known that such a simple picture is in fact appropriate. Specifically, studies of the essential singularity at the coexistence curve displayed in series expansions for the free energy in the  $d = 2$  Ising model (Baker and Kim 1980) have shown that the results can be understood *quantitatively* in terms of a critical droplet with bulk and surface renormalisations of the type suggested above (Harris 1984, Sim and Bruce 1984). Certainly, the picture is intrinsically more plausible in the context of the critical droplet, since the latter is essentially a sphere with a radius which is large compared to the coherence length  $\xi$  of the fluctuations within the boundaries. By contrast the anisotropic droplets which, it is envisaged, dominate the two-site overlap probability do *not* in general have dimensions large on the scale of  $\xi$ . However in  $d = 2$ , the typical transverse dimension ( $\sim r^{1/2}\xi^{1/2}$ ) is large compared to  $\xi$  (in the  $r \gg \xi$  regime of interest), enhancing the plausibility of the picture proposed here in the  $d = 2$  case at least.

We turn, now, to the interrelated question of the  $q$ -dependence of the correlation function suggested by equation (3.5a). Clearly the inference is that, apart from the explicit factor of  $q$ , the entire  $q$ -dependence of the correlation function (at large distances) can be subsumed into the  $q$ -dependence of the order parameter and the  $q$ -dependence of the correlation length, or equivalently the surface tension. Strictly (cf Schmittmann and Bruce 1984) the isolated interface theory underpinning (3.5a) does not predict *any*  $q$ -dependence in the correlation length, presumably because it fails to capture the dressing of a nominally isolated interface separating two phases by droplets of one of the  $q - 2$  other phases, as observed in Monte Carlo studies (Selke and Pesch 1982). However, the theory also fails to incorporate overhang effects which must also in general contribute to interfacial properties. Thus the suggestion that the  $q$ -dependence implied by equation (3.5a) should in fact be taken at face value does not appear to be a significantly stronger proposition than that already made: in effect the philosophy is that the structure of the equation is trustworthy even if the theory is unable to handle the details of the interfacial properties correctly. Nevertheless it



is in the context of the implicit  $q$ -dependence that the 'strong' view of equation (3.5a) might be put most directly to test. Specifically, in the  $q \rightarrow 1$  limit the pair correlation function defined in equation (2.4) gives the probability that two sites a distance  $r$  apart belong to the same cluster (finite or infinite) in the (bond) percolation problem, the order parameter  $Q$  then giving the probability that a site belongs to the infinite cluster. Equation (3.5a) taken in conjunction with the known form of the ordered phase correlation function for the  $d = 2$  Ising ( $q = 2$ ) problem would then offer a quite explicit prediction for the behaviour of this, the percolation pair-connectedness function. Furthermore, if the relationship (1.2) does indeed hold for general  $q$  there follows immediately a unique link between the length  $\xi$  which sets the scale on which the pair connectedness function decays and the typical cluster size setting the scale on which the cluster number distribution decays (Schmittmann and Bruce 1984).

Next, let us review the specific  $d = 2$  realisation of equation (3.5a), explored in § 3.3. As we have already emphasised, neither the results nor the basic philosophy of these calculations is new. Abraham (1983) has presented a very similar analysis of the  $d = 2$  correlation function: he constructs the relevant droplets ('bubbles') from two solid-on-solid interfaces, treated with transfer matrix methods similar to those used in the appendix. Fisher (1984) shows that the essential structure of the correlation function can be understood by modelling the two relevant interfaces with the paths traversed by two walkers at large on a  $(1+1)$ -dimensional lattice, moving uniformly in time (the direction of the droplet axis) but randomly in space, and destined to annihilate on intersection. In any event the picture which emerges from these various studies offers a simple and compelling interpretation of the behaviour of the  $d = 2$  Ising correlation function. The relationship (1.2) is a simple reflection of the fact that it requires *two* interfaces of length  $r$  to produce the ('minimal') droplet contributing to the correlation function. The anomalous  $r^{-2}$  prefactor originates in the entropy-driven wandering of two interfaces which cannot cross.

In fact, the origins of the  $r$ -dependent prefactor in  $\hat{g}(r, \xi)$  and the sense in which it is 'anomalous' merit closer scrutiny. To motivate this final discussion topic it is relevant to record at this point that this prefactor actually plays a key role in determining the nature of the wavevector dependent susceptibility  $\chi(k, \xi)$  (the Fourier transform of  $\hat{g}(r, \xi)$ ), which is measurable by inelastic neutron or x-ray scattering studies. In particular, the form of the prefactor in the direct space correlation function (at large  $r$ ) determines the singularity of the Fourier transform lying closest to the origin in the complex wavevector plane. Thus, while an Ornstein-Zernike ( $r^{-(d-1)/2}$ ) prefactor manifests itself in a simple two-pole structure in  $\chi(k, \xi)$ , the  $r^{-2}$  prefactor appropriate in  $d = 2$  for  $T < T_c$  implies two square root branch points. Combining this information with a knowledge of the large- $k$  behaviour of  $\chi(k, \xi)$ , in  $d = 2$ , Tarko and Fisher (1975) devised approximants for  $\chi(k, \xi)$  over the entire scaling regime. Subsequently Tracy and McCoy (1975) computed the *exact* form of the function  $\chi(k, \xi)$  for the  $d = 2$  Ising model; the more refined of the two Tarko-Fisher approximants was found to reproduce the exact behaviour of the scaling function to within 3%. Recently Cowley *et al* (1984) have studied the inelastic neutron scattering cross section in the quasi-two-dimensional antiferromagnet  $K_2CoF_4$ . The results obtained in the ordered phase can be very satisfactorily parameterised with the (latter) Tarko-Fisher form; attempts to parameterise the results with the Ornstein-Zernike scaling function lead to marked inconsistencies. The moral is clear: the form of the subdominant power law prefactor in the large distance behaviour of  $\hat{g}(r, \xi)$  has distinctive observable consequences for the form of  $\chi(k, \xi)$ .

Pursuing, then, the physics of this prefactor we consider first the high-temperature phase. Here, the Ornstein–Zernike prefactor (which appears to be universally appropriate for  $T > T_c$ ) can be understood within a string framework (§ 3.3.2; Stack and Stone 1981): essentially, each of the  $d - 1$  zero modes of the string (fluctuating modes costing zero energy, modulo boundary effects) contributes a factor of  $r^{-1/2}$  to the prefactor. In the low-temperature phase the physics of the correlation function is quite different: the two-point function is controlled not by strings connecting the two points but by droplets embracing the two points. It is tempting to anticipate that the correlation function prefactor will then be controlled by the zero modes of the relevant droplets. However, in contrast to the situation above the critical point, zero-mode counting alone is insufficient to prescribe the prefactor completely. Certainly, in the  $d = 2$  case studied here one factor of  $r^{-1/2}$  may be traced to each of the two zero modes associated with the translation (centre of mass motion) and the breathing (relative motion) of the contributing anisotropic droplets. However the prefactor contains a *further* power of  $r^{-1}$  (the ‘death factor’ discussed by Fisher (1984)) which originates in the fact that the only configurations which actually contribute are those in which the two wildly wandering interfaces do not meet. It seems clear that the latter effect is a pathology peculiar to the  $d = 2$  case (and that the prefactor is ‘anomalous’ in this sense): the large amplitude wandering of the  $d = 2$  interfaces is a reflection of the relatively low interfacial energy cost of such fluctuations. In  $d = 3$  the typical relevant droplet is (cf our discussion in § 3.2) likely to be essentially cylindrical with a radius of the order of  $\xi$ , the wandering of the surface being strongly controlled by the energy of the ‘cylindrical’ portion of the interface. In this situation there are three relevant modes which are not mechanically stable—two associated with translation and one with breathing (radial distortion) of the droplet. Moreover, while there is likely to be nothing to correspond to the two-dimensional ‘death factor’, it does appear that there will be additional powers of  $r$  reflecting the way in which the sum over all such configurations is cut off. (In  $d = 2$  the cutoff is controlled simply by the configurational weight of the ‘end’ strings and is  $r$ -independent; the  $d = 2$  case is ‘anomalous’ in this sense also.)

Clearly, the  $d = 3$  problem requires rather careful consideration. Certainly, low-temperature transfer-matrix studies (Fisher and Camp 1971, Camp 1973) suggest that an Ornstein–Zernike prefactor should be appropriate in the ordered phase for all  $d \geq 3$ . However these same studies, if taken at face value, would also indicate that the failure of the Ornstein–Zernike theory in the  $d = 2$  case is a pathology peculiar to the *nearest-neighbour* lattice Ising model (and that the Ornstein–Zernike form is restored, for example, by next-nearest-neighbour interactions). With the perspectives of the present work and that of Abraham (1983) and Fisher (1984) it now seems clear that (as was speculated by Fisher and Camp (1971)) this is actually not true: the failure of the Ornstein–Zernike theory for  $d = 2$ ,  $T < T_c$  is, it would appear, a universal feature of Ising (scalar order parameter) systems. Accordingly it seems that the available evidence does not rule out the possibility that the failure of the Ornstein–Zernike theory is actually generic to ordered phases.

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**Appendix. Transfer operator treatment of the two-string partition function**

We outline here how one may use transfer operator methods to treat the configurational integral (3.19b). In fact we shall consider the rather more general situation in which a string is pinned so that its two ends lie, respectively, distances  $\tilde{y}_1$  and  $\tilde{y}_2$  from a repulsive potential barrier  $\tilde{V}$ . The required functional integral is then

$$Z_{2R}(r, \tilde{y}_1, \tilde{y}_2) \approx \int Df \exp \left\{ -\sigma_0 r - \int_0^r dx \left[ \frac{1}{2} \sigma_0 (\nabla f(x))^2 + \tilde{V}(f(x)) \right] \right\} \tag{A1}$$

where the boundary conditions are

$$f(0) = \tilde{y}_1 \quad f(r) = \tilde{y}_2. \tag{A2}$$

Discretising the problem and adopting a measure consistent with (3.14) we find

$$Z_{2R}(r, \tilde{y}_1, \tilde{y}_2) = a_0^{-(N-1)} \int \prod_{n=1}^{N-1} df(x_n) \exp \left( -\sigma_0 r - \frac{\sigma_0}{2a_0} \sum_{n=1}^N (f(x_n) - f(x_{n-1}))^2 - a_0 \sum_{n=1}^{N-1} \tilde{V}(f(x_n)) \right). \tag{A3}$$

The transfer operator method then shows that

$$Z_{2R}(r, \tilde{y}_1, \tilde{y}_2) = a_0 \exp(-\sigma_0 r) \int dk \left( \frac{\lambda_k}{a_0} \right)^N \phi_k(\tilde{y}_1) \phi_k^*(\tilde{y}_2) \tag{A4}$$

where  $\lambda_k$  and  $\phi_k$  are eigenvalues and eigenfunctions of a transfer integral equation which (in the  $a_0 \rightarrow 0$  limit) may be recast in the pseudo-Schrödinger form (see e.g. Scalapino *et al* 1972)

$$\left( a_0 \tilde{V}(u) - \frac{a_0}{2\sigma_0} \frac{d^2}{du^2} \right) \phi_k(u) = \varepsilon_k \phi_k(u) \tag{A5}$$

with

$$\varepsilon_k = \frac{1}{2} \ln(2\pi a_0 / \sigma_0) - \ln \lambda_k. \tag{A6}$$

The behaviour of (A4) has been investigated for a number of different forms of repulsive potential  $\tilde{V}$ ; the essential results are independent of the form. Here we choose simply

$$\tilde{V} = w\delta(u). \tag{A7}$$

One then readily finds

$$Z_{2R}(r, \tilde{y}_1, \tilde{y}_2) = \left( \frac{a_0}{2\pi} \right) \exp \left[ -\sigma_0 r - \frac{r}{2a_0} \ln \left( \frac{\sigma_0 a_0}{2\pi} \right) \right] \int_0^\infty \left[ \frac{dk e^{-rk^2/2\sigma_0}}{1 + w^2 \sigma_0^2 / k^2} \right] \times \left[ 1 + \left( 1 + \frac{2w^2 \sigma_0^2}{k^2} \right) \cos[k(\tilde{y}_1 - \tilde{y}_2)] - \frac{2w\sigma_0}{k} \sin[k(\tilde{y}_1 + \tilde{y}_2)] - \frac{2w^2 \sigma_0^2}{k^2} \cos[k(\tilde{y}_1 + \tilde{y}_2)] \right]. \tag{A8}$$

The single free-string result is obtained by setting  $\tilde{y}_1 = \tilde{y}_2 = w = 0$  yielding (cf (3.17) with  $d = 2$ )

$$Z_{1S}(r) = (\sigma a_0^2 / 2\pi r)^{1/2} e^{-\sigma r} \tag{A9}$$

where we have adopted the same licence with respect to the fluctuation renormalisation of the string tension as in the text; the specific details of that renormalisation differ

from that indicated in equation (3.17), as one would expect in view of the differences of detail in the short wavelength behaviour inherent in the two calculations.

In the limit of large  $r$  (or, equivalently, in the limit of a large potential strength  $w$ ) where  $w^2\sigma_0r \gg 1$  (A8) yields

$$Z_{2R}(r, \tilde{y}_1, \tilde{y}_2) = \left(\frac{\sigma_0 a_0^2}{2\pi r}\right)^{1/2} \exp\left[-\sigma_0 r - \frac{r}{2a_0} \ln\left(\frac{\sigma_0 a_0}{2\pi}\right)\right] \\ \times \{\exp[-\sigma_0(\tilde{y}_1 - \tilde{y}_2)^2/2r] - \exp[-\sigma_0(\tilde{y}_1 + \tilde{y}_2)^2/2r]\}. \quad (\text{A10})$$

For  $\tilde{y}_1 = \tilde{y}_2 = \tilde{y}$  we find

$$Z_{2R}(r, \tilde{y}) = Z_{1S}(r)\{1 - \exp(-2\sigma_0\tilde{y}^2/r)\} \quad (\text{A11})$$

which coincides (to within fluctuation renormalisations of  $\sigma$ ) with equation (3.27) of the text.

Finally, in the limit in which the  $\tilde{y}$  variables are of the order of  $\xi$  or less we find

$$Z_{2R}(r, \tilde{y}_1, \tilde{y}_2) \approx (2\sigma\tilde{y}_1\tilde{y}_2/r)Z_{1S}(r) \quad (\text{A12})$$

which implies for the generalisation of equation (3.19a)

$$Z_{2S}(r, y_1, y_2) \approx (\sigma y_1 y_2/r)Z_{1S}^2(r) \quad (\text{A13})$$

thus recovering the result quoted in equation (3.28b).

*Note added in proof.* In a recent paper Abraham *et al* (1984) have shown that a theory of the three-dimensional ordered phase correlation function, based upon the statistical mechanics of random cylinders, does yield Ornstein-Zernike behaviour at low enough temperatures.

## References

- Abraham D B 1983 *Phys. Rev. Lett.* **50** 291  
 Abraham D B, Chayes J T and Chayes L 1984 *Harvard University Preprint*  
 Baker G A Jr and Kim D 1980 *J. Phys. A: Math. Gen.* **13** L103  
 Bruce A D and Wallace DJ 1983 *J. Phys. A: Math. Gen.* **16** 1721  
 Camp W J 1973 *Phys. Rev.* **B 7** 3187  
 Camp W J and Fisher M E 1972 *Phys. Rev.* **B 6** 946  
 Cowley R A, Hagen M, Belanger D and Ikeda H 1984 *J. Phys. C: Solid State Phys.* **17** 3673  
 Fisher M E 1967 *Physics* **3** 255  
 — 1984 *J. Stat. Phys.* **34** 667  
 Fisher M E and Burford R J 1967 *Phys. Rev.* **156** 583  
 Fisher M E and Camp W J 1971 *Phys. Rev. Lett.* **26** 565  
 Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 536  
 Harris C K 1984 *J. Phys. A: Math. Gen.* **17** 143  
 McCoy B M and Wu T T 1973 *The two dimensional Ising Model* (Harvard: Harvard University Press)  
 Reatto L and Rastelli E 1972 *J. Phys. C: Solid State Phys.* **5** 2785  
 Scalapino D J, Sears M and Ferrell R S 1972 *Phys. Rev.* **B 6** 3409  
 Schmittmann B 1982 *J. Phys. A: Math. Gen.* **15** 3571  
 Schmittmann B and Bruce A D 1984 *J. Phys. A: Math. Gen.* submitted for publication  
 Selke W and Pesch W 1982 *Z. Phys.* **B 47** 335  
 Sim J and Bruce A D 1984 *J. Phys. A: Math. Gen.* submitted for publication  
 Stack J D and Stone M 1981 *Phys. Lett.* **100B** 476  
 Stauffer D 1978 *Z. Phys.* **B 30** 173  
 Tarko H B and Fisher M E 1975 *Phys. Rev. B* **11** 1217  
 Tracy C A and McCoy B M 1975 *Phys. Rev. B* **12** 368  
 Widom B 1972 in *Phase Transitions and Critical Phenomena* vol 2 ed C Domb and M S Green (New York: Academic) p 79  
 Wu T T 1966 *Phys. Rev.* **149** 380